A Characterisation of $G_2(K)$

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Abstract

This paper gives a characterization of the group $G_2(K)$ over some algebraically closed field K of characteristic not 2 inside the class of simple K^*-groups of finite Morley rank not interpreting a bad field using the structure of centralizers of involutions. This implies a general characterization of tame K^*-groups of odd type whose centralizers have a certain natural structure.

1 Introduction

This paper belongs to a series of publications on the classification of of tame simple groups of finite Morley rank and odd type. The result and the methods used to achieve it are intimatly related to the characterisation of $PSp(4, K)$ given [4]. The motivation for this problem and all necessary definitions can be found there. The underlying conjecture is the following.

Conjecture 1 *Let G be a simple tame K^*-group of odd type and Prüfer 2-rank 2. Let $i \in G$ be any involution and $C := C_G(i)^\circ/O(C_G(i))$.*

(1) If $C \cong \mathrm{GL}_2(K)$, then $G \cong \mathrm{PSL}_3(K)$

(2) If $C \cong \mathrm{PSL}_2(K) \times K^*$, then $G \cong \mathrm{PSp}_4(K)$.

(3) If $C \cong \mathrm{SL}_2(K) * \mathrm{SL}_2(K)$, where the two copies of $\mathrm{SL}_2(K)$ intersect non-trivially, then $G \cong \mathrm{PSp}_4(K)$ or $G \cong \mathrm{G}_2(K)$.

Furthermore one of these three cases holds.

We are going to prove cases (2) and (3) of the conjecture using [4]. In [6] one can furthermore find a partial result of case (1), if $O(C_G(i)) = 1$. To prove the complete case (1) seems to be extremely challenging and one will probably need new methods. Especially we prove the following theorem:

Theorem 1 *Let G be a simple K^*-group of finite Morley rank that does not interpret a bad field. Assume that G contains an involution i, such that either*

(i) $C_G(i)^\circ/O(C_G(i)) \cong \mathrm{SL}_2(K) * \mathrm{SL}_2(K)$ *where the two copies of* $\mathrm{SL}(2, K)$ *intersect non-trivially or*

(ii) $C_G(i)^\circ/O(C_G(i)) \cong \mathrm{PSL}(2, K) \times K^*$

for an algebraically closed field K of characteristic neither 2 nor 3. Then $G \cong \mathrm{PSp}_4(K)$ *or* $G \cong \mathrm{G}_2(K)$.

2 Basic Results

We are first going to show, that case (ii) of Theorem 1 implies case (i).

Proposition 2 *Let G be a simple K^*-group of odd type that does not interpret any bad field. Assume that $pr(G) = 2$ and let D be the four-subgroup which is contained in the connected component of a Sylow 2-subgroup S of G. Set $O_D := \bigcap_{l \in D^*} O(C_G(l))$. If $T_0 := C_G(D)^\circ/O_D \cong K^* \times K^*$ for some algebraically closed field of characteristic not 2, then $C_G(i)^\circ/O(C_G(i))$ is isomorphic to one of the following groups for any $i \in D^*$.*

(i) $K^* \times K^*$.

(ii) $\mathrm{GL}_2(K)$

(iii) $\mathrm{PSL}_2(K) \times K^* \cong \mathrm{GL}_2(K)/\langle -I \rangle$ *where* $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

(iv) $\mathrm{SL}_2(K) * \mathrm{SL}_2(K)$ *where the two copies of* $\mathrm{SL}_2(K)$ *intersect non-trivially.*

If $C_G(i)^\circ/O(C_G(i))$ is isomorphic to one of the groups in (iii) or (iv), then $O(C_G(l)) = 1$ for all $l \in D^*$.

Proof. Let $i \in D^*$, then $C := C_G(i)^\circ/O(C_G(i))$ is a central product of an abelian divisible group T and a semisimple group H all of whose components are simple algebraic groups over algebraically closed fields of characteristic different from 2 by [9]. Furthermore $pr(C) = 2$ by [5] and $pr(H) = pr(C) - pr(T)$ by Lemma [5]. Thus H is the central product of simple algebraic groups of Prüfer 2-rank less than or equal to 2 by [11, 27.5].

Set $C_k := C_G(k)$ and $O_k := O(C_k)$ for all $k \in I(G)$. Let θ be the signalizer functor defined by $\theta(s) := O_s$ for all $s \in I(G)$. Then $O_i \cap C_j = O_j \cap C_i = O_j \cap O_i = O_D$ for all $j \in D^*$, $j \neq i$. Hence

$$C_i \cap C_j/O_D \cong (C_i \cap C_j)O_i/O_i \cong (C_i \cap C_j)O_j/O_j.$$

Thus by [2, 2.52] $(C_i \cap C_j)/O_D \cong C_{C_i/O_i}(jO_i) \cong C_{C_j/O_j}(iO_j)$. This implies that $T \leq C_C(jO_i) \cong C_G(D)/O_D = T_0$ and either $T = 1$, $T \cong K^*$ or $T \cong T_0$, since T is connected and G does not interpret a bad field.

Thus (i) holds, if $pr(T) = 2$, as in this case $H = 1$ and $T \cong T_0$. If $pr(T) = 1$, then $T \cong K^*$ and $pr(H) = 1$. Hence H is of type $A_1(K)$, i.e. $H \cong \mathrm{SL}_2(K)$ or $H \cong \mathrm{PSL}_2(K)$. If $C = H \times T$, then $H \cong \mathrm{PSL}_2(K)$, as $C_C(jO_i) \cong T_0$ for any involution $j \in H$. Thus $C \cong \mathrm{GL}_2(K)/\langle -I \rangle$ in this case. If on the other hand $C = H * T$, where $H \cap T$ is non-trivial, then $H \cong \mathrm{SL}_2(K)$ and $C \cong \mathrm{GL}_2(K)$.

If finally $pr(H) = 2$, then T is trivial by [5] and H is of type $A_1(K) \times A_1(K)$, $A_2(K)$, $C_2(K)$ or $\mathrm{G}_2(K)$. However, H has to contain a central involution. Thus H cannot be of type $A_2(K)$ or $\mathrm{G}_2(K)$ and it cannot be isomorphic to $\mathrm{PSp}_4(K)$. As furthermore $C_C(jO_i) \cong T_0$ for any $j \in D \backslash \langle i \rangle$, H cannot be isomorphic to $\mathrm{Sp}_4(K)$ either. Hence H is of type $A_1(K) \times A_1(K)$, contains the central involution i and $C_C(jO_i) \cong T_0$ for any other involution j. Especially $C \cong \mathrm{SL}_2(K) * \mathrm{SL}_2(K)$, where the two copies of $\mathrm{SL}_2(K)$ intersect in $\langle i \rangle$.

If C is isomorphic to the group in (iii) or (iv), then $C_G(i)^\circ$ contains an elementary abelian subgroup E of order 8 which contains D by [4] as C does. Thus $O(C_G(i)) = 1$ by [5, 17]. □

Lemma 3 *Let G be a K^*-group of finite Morley rank. Assume that there exist two definable subgroups $N, H \leq G$ such that $N \lhd H$ and $\overline{H} := H/N$ is a proper simple section of G allowing no graph automorphisms. Then $N_G(H) = C_{N_G(H)}(\overline{H})H$.*

Proof. Let $g \in N_G(H)$ and set $R := d(g)$. $R \leq N_G(H)$ acts on $\overline{H}$. As G is a K^*-group and H is a proper simple section, H is a simple algebraic group over an algebraically closed field. Consider the semidirect product $\overline{H} \rtimes R/C_R(\overline{H})$. Then, viewing $R/C_R(\overline{H})$ as a subgroup of $Aut(\overline{H})$, $R/C_R(\overline{H}) \leq Inn(\overline{H})\Gamma$, where Γ are the graph automorphisms of $\overline{H}$ by [10, 8.4]. As $\Gamma = 1$, either $g \in C_{N_G(H)}(\overline{H})$ or there exists an $h \in H$ such that $x^g \in x^h N$ for all $x \in H$ and $gh^{-1} \in C_{N_G(H)}(\overline{H})$. Thus $g \in C_{N_G(H)}(\overline{H})H$. □

Lemma 4 *Let H be a group of finite Morley rank, such that H° is a reductive algebraic group and let $\overline{H}^\circ := H^\circ/Z(H^\circ)$. Then $C_{C_H(Z(H^\circ))}(H^\circ) = C_{C_H(Z(H^\circ))}(\overline{H}^\circ)$.*

Proof. Let $g \in C_H(\overline{H}^\circ) \cap C_H(Z(H^\circ))$. As H° is reductive, $\overline{H}^\circ$ is a simple group by [11, 27.5] and $m := o(\overline{g}) < |H/H^\circ|$. Set $z_x := x^{-1}x^g \in Z(H^\circ)$ for all $x \in H^\circ$. Then, as $x^g = xz_x$, $x = x^{g^m} = xz_x^m$ and $o(z_x)|m$ for all $x \in H^\circ$. On the other hand $(x^m)^g = (xz_x)^m = x^m z_x^m$ and $g \in C_H(x^m)$ for all $x \in H^\circ$. Especially $C_P(g) = P$, for any maximal torus P of H°, since maximal tori are divisible. As finally H° is a reductive algebraic group, $H^\circ = \langle P^h |\ h \in H^\circ \rangle \leq C_H(g)$ by [11, ex. 12, p. 162] and $g \in C_H(H^\circ)$. □

Theorem 5 *Let G be a simple K^*-group of finite Morley rank that does not interpret a bad field. Assume that $C_G(i)^\circ/O(C_G(i)) \cong \mathrm{PSL}_2(K) \times K^*$ for an*

involution $i \in G$ where K is an algebraically closed field of characteristic not 2. Then G contains an involution j, such that $C_G(j)^\circ \cong \mathrm{SL}_2(K) * \mathrm{SL}_2(K)$.

Proof. Let S be Sylow 2-subgroup of G that contains i and $D \leq S^\circ$ a four-subgroup. We may assume that $i \in D$ by Proposition 2. Set $T := C_G(D)^\circ$. Hence $O(C_G(t)) = 1$ for all $t \in D^*$ by Proposition 2 again. Furthermore $G = \langle C_G(k)^\circ |\ k \in D_1^* \rangle$ for any four-subgroup $D_1 \leq G$ by [5]. Set $T := C_G(D)^\circ$ and let $u \in (C_G(D) \cap C_G(i)^\circ) \backslash T$. Then $C_T(u) = Z(C_G(i)^\circ)$.

Assume that $C_G(j)^\circ \cong \mathrm{GL}_2(K)$ for some $j \in D^*$. Then there exists an involution $w \in C_G(j)^\circ \cap N_G(D)$ such that $i^w = ij$. Then $uu^w = w^u w \in C_G(D) \cap C_G(j)^\circ = T$. Furthermore, as u centralises $Z(C_G(i)^\circ)$, u^w centralises $Z(C_G(i)^\circ)$ as well. On the other hand $C_T(u^w) = C_T(u)^w = Z(C_G(ij)^\circ)$. Contradiction as $Z(C_G(ij)^\circ) \neq Z(C_G(i)^\circ)$.

Assume that $C_G(j)^\circ \cong \mathrm{PSL}_2(K) \times K^*$ for some $j \in D^*$, $j \neq i$. Let $v \in C_{C_G(j)^\circ}(D) \backslash T$ be an involution. As $C_T(v)^\circ = Z(C_G(j)^\circ)$, $v \in C_G(i) \backslash C_G(i)^\circ$. Let $C := C_G(i)^\circ$ and $\overline{C} := C/Z(C)$. By Lemma 3 $C_G(i) = C_{C_G(i)}(\overline{C})C$.

v normalises $Z(C) \cong K^*$ and G does not interpret a bad field. Thus v either inverts $Z(C)$ or centralizes it by [10, 10.5]. Since $C_T(v) = Z(C_G(j)^\circ)$, the second case cannot occur and v inverts $Z(C)$. Let $x \in C$, such that $xv \in C_{C_G(i)}(\overline{C})$. Then $x \in N_G(D) \cap C = T\langle u \rangle$ and xv inverts $Z(C)$. As u inverts $Z(C_G(j)^\circ)$ as above, yv inverts T for all $y \in uT$. Hence $x \in T$, since $xv \in C_{C_G(i)}(\overline{C})$. Furthermore we may assume that $x \in Z(C_G(j)^\circ)$, as $T = Z(C_G(j)^\circ)Z(C)$. Thus $(xv)^2 = x^2 \in C_C(\overline{C}) = Z(C) \cap Z(C_G(j)^\circ) = 1$ by Lemma 4 and $x \in \langle j \rangle$. Then $[xv, C] \subseteq Z(C)$ and $C = Z(C) * C_C(xv)$ by [10, ex. 10, p. 98] where $Z(C) \cap C_C(xv) = \langle i \rangle$. Set $H := C_C(xv)^\circ$. As $C \cong \mathrm{PSL}_2(K) \times K^*$, $H \cong \mathrm{PSL}_2(K)$ and $C = Z(C) \times H$. As $x \in \langle j \rangle$, $xv \in I(C_G(D))$. Thus $D_1 := \langle i, xv \rangle$ is a four-subgroup and $G := \langle C_G(k) |\ k \in D_1^* \rangle$. Furthermore $H \leq C_G(D_1)$. As H is a normal subgroup of $C_G(i)^\circ$ and G is simple, H cannot be a normal subgroup of both $C_G(ux)$ and $C_G(uxi)$. As ux and uxi are conjugate to involutions in D by elementary computation in $\mathrm{GL}_2(K)/ < -I >$, this implies that $C_G(k) \cong \mathrm{SL}(2, K) * \mathrm{SL}(2, K)$ for $k = ux$ or $k = uxi$ by Proposition 2.

As finally $G = \langle C_G(k)^\circ |\ k \in D^* \rangle$, it is impossible that $C_G(k)^\circ$ is abelian for all $k \in D \backslash \langle i \rangle$. This implies the claim by Proposition 2. □

Thus Theorem 1 basically consists of two parts. The first one is the characterization of PSp_4 as in [4].

Fact 6 *Let C_1 and C_2 be the non-isomorphic centralizers in $\mathrm{PSp}_4(K)$ of two involutions where K is an algebraically closed field of characteristic not 2. Let G be a K^*-group of finite Morley rank that does not interpret a bad field. Assume that G contains two involutions i and j such that $C_G(i) \cong C_1$ and $C_G(j) \cong C_2$. Then $G \cong \mathrm{PSp}_4(K)$.*

The corresponding result for $\mathrm{G}_2(K)$ is the following theorem, which will be proven in the following section.

Theorem 7 *Let C be the centralizer in $\mathrm{G}_2(K)$ of an involutions where K is an algebraically closed field of characteristic not 2. Let G be a K^*-group of finite Morley rank that does not interpret a bad field. Assume that G contains one conjugacy class i^G of involutions such that $C_G(i) \cong C$. Then $G \cong \mathrm{G}_2(K)$ if $char(K) \neq 3$.*

3 A characterization of $\mathrm{G}_2(K)$

In this section we prove Theorem 7. Let G be as in Theorem 7, $D := \langle i_0, i_1 \rangle$ a four-subgroup which is contained in the connected component of a Sylow 2-subgroup of G and set $i_2 = i_0 i_1$. Let furthermore $T := C_G(D)^\circ$. $C_G(i_0)$ contains four quasiunipotent subgroups which are normalized by T and isomorphic to K^+. Let X and Y be two of them that centralize each other and let v be an involution in $N_{C_G(i_0)}(T) \backslash T$ that normalizes X. Write $\overline{x} := xT$ for all $x \in N_G(T)$.

Lemma 8 *Let $W := N_G(T)/T$. Then $|W| = 12$ and W is generated by two involutions $\overline{w}$ and $\overline{v}$, where $w \in I(C_G(i_1))$, such that $i_0^{wv} = i_1 = i_2^{vw}$ and $o(wv) = 6$. The center of W is $\langle \overline{z} \rangle$ where $z := (wv)^3$. Set $y := (wv)^4 = (wv)z$ and $w_{k+1} := w^{y^k}$, $v_k := v^{y^k}$ for $k \in \mathbb{N}$. Then $\overline{w}_k = \overline{w_{k+3}}$ and $\overline{v}_k = \overline{v_{k+3}}$. The elements in W are hence*

$$\overline{1}, \overline{z} = \overline{(wv)^3}, \overline{wv} = \overline{yz}, \overline{(wv)^2} = \overline{y^{-1}}, \overline{(wv)^4} = \overline{y}, \overline{(wv)^5} = \overline{vw} = \overline{y^{-1}z}$$

and

$$\overline{w}_{k+1} = \overline{w^{(vw)^{2k}}} = \overline{y^k w} \text{ and } \overline{v}_k = \overline{v^{(vw)^{2k}}} = \overline{y^k v}$$

for $k = 0, 1, 2$.

Proof. As G contains one conjugacy class of involutions, all involutions of D are conjugate in $N_G(T)$ by [10, 10.22]. $N_G(D)/C_G(D) \cong S_3$, $N_G(T) = N_G(D)$ and $|C_G(D)/C_G(T)| = 2$, which implies that W is the dihedral group of order 12. □

Lemma 9 *Let w_0, v_0 as in Lemma 8. Then $w_0 T X w_0 \subseteq T X w_0 X$ and $v_0 T Y v_0 \subseteq T Y v_0 Y$. Furthermore $C_G(i_0)^\circ = \langle T, X, Y, v_0, w_0 \rangle$.*

Proof. $L := \langle X, T, X^{w_0} \rangle$ is a reductive algebraic group of rank 1. L has thus a BN-pair (B_1, N_1), where $B_1 := TX$ and $N_1 = \langle w_0, T \rangle$. Hence $w_0 T X w_0 \subseteq TX \cup T X w_0 X = T X w_0 X$ and $L := \langle T, X, w_o \rangle$. The same argument for $\langle Y, T, Y_0 \rangle$ yields the remaining part of the lemma. □

Lemma 10 *Let $X_1 := X$, $Y_1 := Y$, $X_2 := X_1^{w_0}$, $Y_2 := Y_1^{v_0}$ and $X_n^{(\lambda)} := X_n^{y^\lambda}$ as well as $Y_n^{(\lambda)} := Y_n^{y^\lambda}$ for any $n = 1, 2$ and $\lambda \in \mathbb{Z}$. Then*

(i) $X_n^{(\lambda)} = X_n^{(\lambda+3)}$ and $Y_n^{(\lambda)} = Y_n^{(\lambda+3)}$ for all $n = 1, 2$ and $\lambda \in \mathbb{Z}$.

(ii) v_λ *centralizes* $X_n^{(\lambda)}$ *and* w_λ *centralizes* $Y_n^{(\lambda)}$ *for* $\lambda = 0, 1, 2$ *and* $n = 1, 2$.

(iii) $(X_1^{(\kappa)})^{w_\lambda} = X_2^{(2\kappa-\lambda)}$ *and* $(X_1^{(\kappa)})^{v_\lambda} = X_1^{(2\kappa-\lambda)}$ *for* $\kappa, \lambda = 0, 1, 2$.

(iv) $(Y_1^{(\kappa)})^{w_\lambda} = Y_1^{(2\kappa-\lambda)}$ *and* $(Y_1^{(\kappa)})^{v_\lambda} = Y_2^{(2\kappa-\lambda)}$ *for* $\kappa, \lambda = 0, 1, 2$.

Proof. Since $\overline{y^3} = \overline{1}$, (i) follows. We have furthermore chosen w_0 such that w_0 centralizes Y_1. Hence v_0 has to centralize X_1 which yields (ii). To prove (iii) note that $\overline{y^\lambda} = \overline{w_\lambda w_0} = \overline{v_\lambda v_0}$ and hence

$$\overline{y^\kappa w_\lambda y^{\lambda-2\kappa} w_0} = \overline{y^{3\kappa-\lambda} w_\lambda w_0} = \overline{y^{-\lambda} y^\lambda} = \overline{1}$$

and

$$\overline{y^\kappa v_\lambda y^{\lambda-2\kappa} v_0} = \overline{y^{3\kappa-\lambda} v_\lambda v_0} = \overline{1}$$

for $\lambda, \kappa \in 0, 1, 2$. Since however $X_1^{w_0} = X_2$, $X_1^{v_0} = X_1$, $Y_1^{w_0} = Y_1$ and $Y_1^{v_0} = Y_2$ by (ii), (iii) and (iv) follow. □

Lemma 11 *Let G be a connected K-group of finite Morley rank such that the solvable radical σ of G is finite. Then G is a central product of quasi-simple algebraic groups over algebraically closed fields.*

Proof. Since any definable action of a definable connected group on a finite set is trivial $\sigma = Z(G)$. Furthermore G/σ is isomorphic to the direct product of simple algebraic groups over algebraically closed fields by [1]. Then $G/\sigma = (G/\sigma)' = G'\sigma/\sigma \cong G'/(G' \cap \sigma)$ and $rk(G) = rk(G')$. As G is connected, $G = G'$ and G is semisimple. Assume that G is quasi-simple. Then G is an algebraic group by [3]. Let now $G/\sigma \cong A_1 \times \cdots \times A_m$ for some $m \in \mathbb{N}$, where A_i are simple algebraic groups over algebraically closed fields for $1 \leq i \leq m$. Let G_i be the preimages of A_i in G for $1 \leq i \leq m$. Then $G = G_1^\circ \ldots G_m^\circ$. Now $[G_i^\circ, G_j^\circ]$ is a connected subgroup of σ and hence trivial for any $1 \leq i < j \leq m$. Furthermore $[G_i^\circ, G_i^\circ] = G_i^\circ$ for any $1 \leq i \leq m$ as above. Thus G is a central product of m quasi-simple algebraic groups by the first case. □

Proposition 12 *If $char(K) \neq 3$ then there exists an element t of order 3, such that $C_G(t)^\circ \cong \mathrm{SL}_3(K)$. Actually $C_G(t)^\circ \cong \mathrm{SL}_3(K)$ for exactly two elements of order 3 in T.*

Proof. Assume that $char(K)$ is not 3. Since $y^3 \in T$, $y = y't$, where we can choose y' to be a 3-element by [10, ex. 11, p. 93]. Let P be a Sylow 3-subgroup of $N_G(T)$. Since $N_G(T)$ is solvable, P is nilpotent-by-finite by [10, 6.20]. Hence there exists an element $t \in T \cap Z(P)$ by [10, ex. 12, p. 14]. It follows that t is centralized by y. Thus $(t^{w_0}t)^{y^2} = (t^{w_0}t)^{w_2 w_0} = t^{y w_0} t = t^{w_0} t$ and $t^{w_0} t \in C_T(y)$. Hence either $t^{w_0} = t^{-1}$ or $t^{w_0} t \in C_T(y)$ is an element of order 3 that is inverted by w_0. We may assume that t is inverted by w_0. Now $C_G(t)^\circ = \langle C_{C_G(t)}(l)^\circ |\ l \in D^* \rangle$ and $C_{C_G(i_0)}(t) = C_{C_G(i_1)}(t)^{y^2} = C_{C_G(i_2)}(t)^y$. However, $C_{C_G(i_0)}(t)^\circ \cong \mathrm{GL}_2(K)$.

We show that the solvable radical σ of $C_G(t)^\circ$ is finite. σ is normalized by T and hence $\sigma^\circ = \langle C_\sigma(l)^\circ | l \in D\rangle$ by [9, 4.12]. However, $C_\sigma(l)^\circ \leq Z(C_{C_G(l)}(t)^\circ)$ for all $l \in D^*$ as $C_\sigma(l)^\circ$ is contained in the solvable radical of $C_{C_G(l)}(t)^\circ$. Thus $\sigma^\circ \leq \bigcap_{l\in D^*} Z(C_{C_G(l)}(t)^\circ) \leq T$. One the other hand $Z(C_{C_G(i_0)}(t)^\circ) \cong K^*$ and as G does not interpret a bad field, σ is either finite or $i_0 \in Z(C_{C_G(i_0)}(t)^\circ) = Z(C_{C_G(i_1)}(t)^\circ)$. The second case cannot occur since $C_G(D)^\circ = T$ and σ finite.

Thus $C_G(t)^\circ$ is a central product of quasi-simple algebraic groups by Lemma 11 and $C_G(t)^\circ \cong \mathrm{SL}_3(K)$. Furthermore $T \leq C_G(t)^\circ$ contains eight elements of order 3 and while y centralizes exactly t and t^{-1}, yw operates transitively on the remaining six. □

The following two propositions can be proven exactly as the corresponding results in [4].

Proposition 13 *There exists a subgroup $V \leq G$ such that V is a maximal quasiunipotent group and is normalized by T.*

Proposition 14 *Let V a maximal quasiunipotent group of G which is normalized by T. Then $N_G(V)^\circ = V \rtimes T$.*

Now we can construct a BN-pair as in [8]

Proposition 15 *If $char(K) \neq 3$ then there exists a maximal quasiunipotent group Q in G which is normalized by T such that*

(i) $Q = V \rtimes Y_1$ where V is normalized by v_0 and

(ii) $Q = M \rtimes X_2^{(1)}$ where M is normalized by w_1.

Furthermore $Q = \langle X_1, X_2^{(1)}, X_2^{(2)}, Y_1, Y_2^{(1)}, Y_1^{(2)}\rangle$.

Proof. Since $char(K)$ is not 3, there exists an element $t \in T$ of order 3, such that $C_G(t)^\circ \cong \mathrm{SL}_3(K)$. We can hence assume that $C_G(t)^\circ$ contains the maximal quasiunipotent subgroup $U := X_1\langle X_2^{(1)}, X_2^{(2)}\rangle$, where X_1 is central in U. Consider $C_G(X_1)$. We proceed as in [4]. $U \leq C_G(X_1)$ and $L := \langle T, Y_1, Y_2\rangle \leq C_G(X_1)$. Thus $C_G(X_1)^\circ$ is not solvable. Furthermore $C_G(X_1)^\circ$ has Prüfer 2-rank 1, since $C_G(T)^\circ = T$. Thus $C_G(X_1)^\circ/\sigma \cong \mathrm{PSL}_2(K)$ by [1] and [5], where σ is the solvable radical of $C_G(X_1)$. On the other hand $L\sigma/\sigma \cong L/(\sigma \cap L) \cong L/Z(L) \cong \mathrm{PSL}_2(K)$. Hence $C_G(X_1)^\circ = L\sigma^\circ$ and $U \leq \sigma$. Set $V := Q(\sigma)$. Then $V^{v_0} = V$.

Assume that $U = V$. Then Y_1U is a quasiunipotent group by [7]. Especially Y_1U is nilpotent and $W := Y_1X_1 < Y_1U$ has infinite index in $N := N_{Y_1U}(W)^\circ$ by [10, 6.3]. N is a quasiunipotent group that is normalized by T and hence $N = \langle C_N(l)^\circ |\ l \in D^*\rangle$ by [9, 4.6]. Furthermore $C_N(i_0) = W$ and $C_N(i_k)$ is

either trivial or equals $X_2^{(k)}$ for $k = 1, 2$ as T normalises N and acts transitively on $X_2\backslash\{1\}$. We may assume that $X_2^{(1)} \leq N$.

Set $P := WX_1^{(2)} \rtimes T$. Then $P \leq N_G(W)$. Let $w \in W\backslash\{1\}$ such that $C_G(w) \cap T = \langle i_0\rangle$. As T acts transitively on $W\backslash\{1\}$, for any element $x \in P$, there exists an element $t_x \in T$ such that $w^x = w^{t_x}$. Thus $P \subseteq C_P(w)T$. As i_1 and i_2 invert W, $C_P(w)$ is a solvable group normalized by D. Thus $C_P(w)^\circ = \langle C_{C_P(w)}(l)^\circ | \, l \in D^*\rangle$ by [10, 4.6]. Furthermore $(C_P(w) \cap C_G(i_0))^\circ = W$, $(C_P(w) \cap C_G(i_1))^\circ \leq X_1^2$ and $(C_P(w) \cap C_G(i_2))^\circ = 1$. Thus $Q := C_P(w)^\circ \leq WX_1^{(2)}$ is a quasiunipotent subgroup of P containing W. Since there is a definable surjective map from $C_P(w) \times T$ onto $N_P(W)$

$$rk\big(WX_1^{(2)}\big) + rk(T) = rk\big(N_P(W)\big) \leq rk\big(C_P(w)\big) + rk(T) = rk(Q) + rk(T).$$

Thus $Q = WX_1^{(2)}$ and $N_P(W) = C_P(w)^\circ \rtimes T$. Especially $Q \leq C_P(w^t)$ for all $t \in T$ and hence $Q \leq C_G(W)$ as T acts transitively on W. Thus $X_2^{(1)} \leq C_G(W)$ and $X_2^{(1)}$ centralizes Y_1. This implies that $Y_1^{(1)} = Y_1^{w_2} \leq C_G(X_2^{(1)})^{w_2} = C_G(X_1)$. As $C_G(X_1)^\circ = L\sigma$ this implies that $Y_1^{(1)} \leq V$. Contradiction.

Hence $U < V$ and either $\langle Y_2^{(1)}, (Y_2^{(1)})^{v_0}\rangle \leq V$ or $\langle Y_1^{(1)}, (Y_1^{(1)})^{v_0}\rangle \leq V$. We may assume that $V = \langle Y_2^{(1)}, Y_1^{(2)}, U\rangle$. Set $Q := Y_1V$. Q is obviously a maximal quasiunipotent subgroup of G and V is normalized by v_0 by construction. Set $Y := \langle Y_1, Y_2^{(1)}, Y_1^{(2)}\rangle$. Then Y is invariant under w_1 and $M := Q \cap Q^{w_1} = \langle Y, X_1X_2^{(2)}\rangle$. Furthermore M is normalized by $X_2^{(1)}$, since $N_Q(M)^\circ = Q$ by [10, 6.3] as in the previous paragraph. □

Theorem 16 *If $char(K) \neq 3$, G is a split BN-pair of Tits rank 2, where $B := N_G(Q)^\circ$, $N := N_G(T)$ and $\overline{w}_2, \overline{v}_0$ are the generators of the Weyl group.*

This proves our result, Theorem 7, by [7], noting that the gap in the proof of the necessary theorem is filled by [12].

Proof. $G = \langle C_G(k) | \, k \in D^*\rangle$ by [5, 18] and $\langle N, X_1Y_1\rangle \geq C_G(i_0)$ by Lemma 9. Thus $G = \langle B, N\rangle$ as i_1, i_2 are conjugate to i_0 in N. BN1 now follows since by Proposition 14 $B = QT$ and thus $B \cap N = T \lhd N$. Furthermore $\langle \overline{v}_0, \overline{w}_1\rangle = N_G(T)/T$ by Lemma 8 proving BN2. Let $S := \{\overline{v}_0, \overline{w}_1\}$ and $W := N/T$.

To prove BN3 we will show that $vBw \subseteq BvwB \cup BvB$ for all $v, w \in N$ such that $\overline{v} \in W$ and $\overline{w} \in S$. By Lemma 10 and Proposition 15

$$1Bv_0 \subseteq B1v_0B$$

$$v_0Bv_0 = v_0TY_1Vv_0 \subseteq TY_1v_0Y_1V \subseteq Bv_0B$$

$$v_2Bv_0 = v_2TY_1Vv_0 = TY_2^{(1)}v_2v_0V \subseteq Bv_2v_0B$$

$$w_0Bv_0 = w_0TY_1Vv_0 = TY_1w_0v_0V \subseteq Bw_0v_0B$$

$$w_1Bv_0 = w_1TY_1Vv_0 = TY_1^{(2)}w_1v_0V \subseteq Bw_1v_0B$$

$$y^2Bv_0 = y^2TY_1Vv_0 = TY_1^{(2)}y^2v_0V \subseteq By^2v_0B$$

$$v_0w_1Bv_0 = v_0w_1TY_1Vv_0 = TY_2^{(1)}v_0w_1v_0 \subseteq Bv_0w_1v_0B$$

and thus

$$zBv_0 = (zv_0)v_0Bv_0 \subseteq w_0Bv_0B \subseteq Bw_0v_0B = BzB$$

$$w_1v_0Bv_0 \subseteq w_1Bv_0B \subseteq Bw_1v_0B$$

$$w_2Bv_0 = (w_2v_0)v_0Bv_0 \subseteq v_0w_1Bv_0B \subseteq Bv_0w_1v_0B = Bw_2B.$$

Finally

$$yBv_0 = yTY_1Vv_0 = TY_1^{(1)}yv_0V = TY_1^{(1)}v_1V = Tv_1Y_1^{(2)}V \subseteq Byv_0B$$

which gives us

$$v_1Bv_0 = (v_1v_0)v_0Bv_0 \subseteq yBv_0B \subseteq Byv_0B = Bv_1B.$$

On the other hand

$$1Bw_1 \subseteq B1w_1B$$

$$w_1Bw_1 = w_1TX_2^{(1)}Mw_1 \subseteq TX_2^{(1)}w_1X_2^{(1)}M \subseteq Bw_1B$$

$$v_1Bw_1 = v_1TX_2^{(1)}Mw_1 = TX_2^{(1)}v_1w_1M \subseteq Bv_1w_1B$$

$$w_2Bw_1 = w_2TX_2^{(1)}Mw_1 = TX_1w_2w_1M \subseteq Bw_2w_1B$$

$$v_0Bw_1 = v_0TX_2^{(1)}Mw_1 = TX_2^{(2)}v_0w_1M \subseteq Bv_0w_1B$$

$$yBw_1 = yTX_2^{(1)}Mw_1 = TX_2^{(2)}yw_1M \subseteq Byw_1B$$

$$w_1v_0Bw_1 = w_1v_0TX_2^{(1)}Mw_1 = TX_1w_1v_0w_1 \subseteq Bw_1v_0w_1B$$

and thus

$$zBw_1 = (zw_1)w_1Bw_1 \subseteq v_1Bw_1B \subseteq Bv_1w_1B = BzB$$

$$v_0 w_1 B w_1 \subseteq v_0 B w_1 B \subseteq B v_0 w_1 B$$

$$w_2 B w_1 = (w_2 w_1) w_1 B w_1 \subseteq y B w_1 B \subseteq B y w_1 B = B w_2 B.$$

Finally

$$y^2 B w_1 = y^2 T X_2^{(1)} M w_1 = T X_2 y^2 w_1 M = T X_2 w_0 M = T w_0 X_1 M \subseteq B y^2 w_1 B$$

which gives us

$$v_2 B w_1 = (v_2 w_1) w_1 B w_1 \subseteq w_1 v_0 B w_1 B \subseteq B w_1 v_0 w_1 B = B v_2 B.$$

and BN3 is valid. By Proposition 15 again $Q^{v_0} = VY_2 \neq VY_1 = Q$ and $Q^{w_2} = MX_1^{(1)} \neq MX_1^{(2)} = Q$ which proves BN4 and gives us the theorem. □

4 Proof of Theorem 1

This section is devoted to the proof that Theorems 6 and 7 imply Theorem 1.

Let G be a group as in Theorem 1. By Theorem 5 there exists an involution $i \in G$, such that $C_G(i)^\circ / O(C_G(i)) \cong L_1 * L_2$ where $L_n \cong \mathrm{SL}_2(K)$ for an algebraically closed field K of characteristic neither 2 nor 3 and $n = 1, 2$. We may assume that $i \in C_G(i)^\circ$ by Lemma 2. Thus $L_1 \cap L_2 = \langle \bar{\imath} \rangle$. Let S be a Sylow 2-subgroup of $C_G(i)$, D a four-subgroup such that $D \leq S^\circ$ and set $T := C_G(D)^\circ$. Then D contains a central involution of S by [10, ex. 12, p. 14] which means that i is central in S, since j and ij are conjugate in S for $j \in D \backslash \langle i \rangle$.

Proposition 17 *$G = \langle C_G(k)^\circ |\ k \in D_1^* \rangle$ for any four-subgroup $D_1 \leq G$ and $O(C_G(k)) = 1$ for all $k \in D^*$.*

Proof. $O(C_G(t)) = 1$ for all $t \in D^*$ by Proposition 2 and $G = \langle C_G(k)^\circ |\ k \in D_1^* \rangle$ for any four-subgroup $D_1 \leq G$ by [5]. □

Corollary 18 *$C_G(k)^\circ$ is nonabelian for all $k \in D^*$.*

Proof. Assume that $C_G(j)^\circ$ is abelian for some $j \in D^*$. Then $C_G(j)^\circ \leq C_G(i)^\circ$ since $i \in C_G(j)^\circ$. As furthermore j and ij are conjugate, $G = C_G(i)^\circ$ by Proposition 17. Contradiction. □

Corollary 19 *Let $j \in D^*$. If i and j are not conjugate in G, then $C_G(j)^\circ \cong \mathrm{PSL}_2(K) \times K^*$.*

Proof. Let $j \in D^*$ be not conjugate to i. Then j is conjugate to ij in $C_G(i)^\circ$. Now $C_G(j)^\circ$ is not abelian by Corollary 18. Furthermore $C_G(D)^\circ \cong K^* \times K^*$ and there are only three possibilities by Proposition 2, namely $C_G(j)^\circ \cong \mathrm{GL}_2(K)$, $C_G(j)^\circ \cong \mathrm{PSL}_2(K) \times K^*$ or $C_G(j)^\circ \cong \mathrm{SL}_2(K) * \mathrm{SL}_2(K)$. The first and third case cannot occur since i and ij are not conjugate, which proves the claim. □

We have to distinguish two different cases:

(i) $C_G(i)$ is connected.

(ii) $C_G(i)$ is not connected.

We are going to show that $G \cong \mathrm{G}_2(K)$ in the first case and $G \cong \mathrm{PSp}_4(K)$ in the second case. Assume from now on that $D = \langle i, j \rangle$.

Lemma 20 *If $C_G(i)$ is connected, then G contains one conjugacy class of involutions.*

Proof. $S \leq C_G(i)$, since $i \in Z(S)$. As Sylow 2-subgroups of G are conjugate, and as $C_G(i)$ contains only two conjugacy classes of involutions i and $j^{C_G(i)}$ – all elements of order 4 are conjugate in $\mathrm{SL}_2(K)$ –, it is enough to show that i and j are conjugate.

Assume that j is not conjugate to i and let $w \in N_G(T)$ such that $D\langle w \rangle$ is an elementary abelian 2-subgroup of order 8. Then $C_G(D) = T\langle w \rangle$ and w inverts T. As i and ij are not conjugate in G, $C_G(j)^\circ \cong \mathrm{PSL}_2(K) \times K^*$ by Corollary 19. Let $u \in (N_G(T) \cap C_G(j)^\circ) \backslash T$ be an involution. Then $u \in C_G(D)$, i.e. since $C_G(D) = T\langle w \rangle$, $u = tw$ for some $t \in T$. Thus u has to invert T. Contradiction as $Z(C_G(j)^\circ) \leq T$ is infinite. □

Corollary 21 *If $C_G(i)$ is connected, then $G \cong \mathrm{G}_2(K)$.*

Proof. If $C_G(i)$ is connected, then G contains one conjugacy class of involutions by Lemma 20. Since furthermore the centralisers of involutions in $\mathrm{G}_2(K)$ are isomorphic to $\mathrm{SL}_2(K) * \mathrm{SL}_2(K)$, the claim follows by Theorem 7. □

Lemma 22 $N_G(L_1) = C_G(i)^\circ C_G(C_G(i)^\circ)$.

Proof. As $C_G(i)^\circ C_G(C_G(i)^\circ) \leq N_G(L_1)$, we only need to prove the reverse inclusion. Let $g \in N_G(L_1)$. As $Z(L_1) = \langle i \rangle$, $g \in C_G(i)$ and $g \in N_G(L_2)$ as well by [10, 7.1]. By Lemma 3 and Lemma 4, $g \in C_G(L_1)L_1 \cap C_G(L_2)L_2$. Thus there exists $l_n \in L_n$ for $n = 1, 2$ such that $gl_n \in C_G(L_n)$. As L_1 and L_2 commute, $gl_1l_2 \in C_G(L_1) \cap C_G(L_2)$ and hence $g \in C_G(C_G(i)^\circ)C_G(i)^\circ$. □

Set $K_s := C_G(C_G(s)^\circ)$ for all involutions $s \in G$.

Lemma 23 *Let $s \in I(G)$. Then $K_s \cap C_G(s)^\circ = Z(C_G(s)^\circ)$. Furthermore*

(i) K_i is a finite group,

(ii) $I(K_i) = i$,

(iii) $K_i \cap K_s = 1$ for all $s \in D^*$ such that $i \neq s$.

Proof. As $Z(C_G(i)^\circ) = \langle i \rangle$, $K_i \leq C_G(i)$ and $K_i \cap C_G(i)^\circ = \langle i \rangle$. Hence $K_i^\circ \leq C_G(i)^\circ \cap K_i = \langle i \rangle$ and K_i is a finite group, proving (i).

To show (ii) let $t \in I(K)$ be an involution. Then $O(C_G(t)) = 1$ by [5]. Assume that $C_G(t)^\circ > C_G(i)^\circ$. Then $C_G(t)^\circ \cong \mathrm{G}_2$ or $C_G(t)^\circ \cong \mathrm{PSp}_4(K)$ by [9]

as $pr(C_G(t)^\circ) = pr(G) = 2$. However this would imply that $C_G(j)^\circ, C_G(ij)^\circ \leq C_G(t)^\circ$ by Proposition 2 and $G = C_G(t)^\circ$ by Proposition 17. Contradiction. Thus $C_G(t)^\circ = C_G(i)^\circ = C_G(it)^\circ$ and $t = i$, as otherwise $\langle i, t\rangle$ is a four-subgroup and $G = C_G(i)^\circ$ by Proposition 17.

Let finally $s \in D^*$ such that $s \neq i$. As $C_G(i)^\circ, C_G(s)^\circ \leq C_G(K_i \cap K_s)$ and si is conjugate to s in $C_G(i)^\circ$, [9, 5.14] and Proposition 17 imply that $C_G(K_s \cap K_t)^\circ = G$. As G is simple, $K_i \cap K_s = 1$. $\square$

Corollary 24 *$C_G(i) = (C_G(i)^\circ * K_i) \rtimes \langle v\rangle$ for any $v \in G$ such that $L_1^v = L_2$. In this case $v^2 \in C_G(i)^\circ K_i$ and we can choose $v \in N_G(T)$ to be a 2-element.*

Proof. Assume that there exists an element $v \in G$ such that $L_1^v = L_2$. As $Z(L_1) = Z(L_2) = \langle i\rangle$, $v \in C_G(i)\backslash C_G(i)^\circ$, $L_2^v = L_1$ by [10, 7.1] and $v^2 \in C_G(i)^\circ K_i$ by Lemma 22.

Let $x \in C_G(i)$. There are two possibilities: Either $x \in N_G(L_1) = C_G(i)^\circ * K_i$ by Lemma 22 and Lemma 23 or $L_1^x = L_2$ and $L_2^x = L_1$ by [10, 7.1]. In the second case $xv \in N_G(L_1) = C_G(i)^\circ K_i$ by Lemma 22 again. Thus $x \in (C_G(i)^\circ * K_i) \rtimes \langle v\rangle$ in all cases.

Let finally $d(v) = V \times F$, where V is a connected divisible group and $F = \langle f\rangle$ a finite cylic group by [10, ex. 10, p. 93]. Then $V \leq C_G(i)^\circ$ and $f^2 \in N_G(L_1)$. Thus f has even order. Let $o(f) = 2^k m$ where $k \geq 1$ and $m \in \mathbb{N}$ is odd. Then f^m is a 2-element such that $L_1^{f^m} = L_2$. Furthermore f^m is contained in a Sylow 2-subgroup of $C_G(i)$ and as they are all conjugate to each other, we may assume that $f^m \in N_G(T)$. $\square$

Lemma 25 *If all involutions in D are conjugate, then $K_i = \langle i\rangle$.*

Proof. Assume that all involutions in D are conjugate. Then there exists an element $y \in N_G(T)$, such that $i^y = j$ by [10, 10.22]. Let $w \in N_G(T) \cap C_G(i)^\circ$ such that $E := D\langle w\rangle$ is an elementary abelian 2-subgroup of order 8. w inverts T and all other involutions that invert T are contained in $wC_G(T) = wTK_i$ by Corollary 24. However, $I(wTK_i) = wT$ by Lemma 23 and $w^y \in wT$. K_i acts hence on $C_G(j)^\circ$ centralising $T\langle w\rangle$. Thus $K_i \leq C_G(j)^\circ K_j$ by Corollary 24. Let $k \in K_i$ and $x \in C_G(j)^\circ$, $k_1 \in K_j$ such that $k = xk_1$. Then $x = kk_1^{-1} \in C_G(E) \cap C_G(j)^\circ = E$. Thus $K_j \leq EK_i$. Furthermore $K_{ij} = K_j^{w_1} \leq EK_i$, where $w_1 \in L_1 \cap N_G(E)$ is an involution conjugating j and ij. Let now $k := |K_i|$. As all involutions in D are conjugate, $k = |K_l|$ for all $l \in D^*$. Furthermore $|EK_i| = 4k$ as $E \cap K_i = \langle i\rangle$. On the other hand $K_i, K_j, K_{ij} \leq EK_i$ and all three subgroups normalise each other. Hence $k^3 \leq 4k$ by Lemma 23 and $k = 2$. Thus $K = \langle i\rangle$, if all involutions in D are conjugate. $\square$

Proposition 26 *If all involutions in D are conjugate, then $C_G(i)$ is connected.*

Proof. Suppose that all involutions in D are conjugate. Then $K_i = \langle i\rangle$ by Lemma 25. Assume that $C_G(i)$ is not connected. Then there exists a 2-element $v \in N_G(T)$ such that $C_G(i) = C_G(i)^\circ \rtimes \langle v\rangle$ where $L_1^v = L_2$ by Lemma 24. As $v^2 \in C_G(i)^\circ$ is a 2-element, it is contained in a maximal torus of $C_G(i)^\circ$. As

maximal tori are divisible, there exists an element $t \in C_G(i)^\circ$, such that $t^2 = v^2$. Assume that $t = t_1 t_2^v$, where $t_1, t_2 \in L_1$. Then

$$(*) \quad 1 = v^2 t^{-2} = v^2 (t_2^{-2v} t_1^{-2}) = v t_2^{-2} v t_1^{-2}.$$

On the other hand $v^2 \in C_G(v)$ and thus $t_1^2 t_2^{2v} = (t_1^2 t_2^{2v})^v = t_2^{2v^2} t_1^{2v}$. This implies that

$$t_2^{-2v^2} t_1^2 = t_1^{2v} t_2^{-2v} \in L_1 \cap L_2 = \langle i \rangle$$

Hence either $t_1^2 = t_2^2$ or $t_1^2 = i t_2^2$. Set $s := t_1^2$. If $t^2 = s s^v$, then $(*)$ implies that vs^{-1} is an involution. If $t^2 = iss^v$, then $(vs^{-1})^2 = i$. By replacing v with a conjugate of vs^{-1}, we may assume that $v \in N_G(T)$, such that $v^2 \in \langle i \rangle$.

As all involutions in D are conjugate and j is conjugate to ij in $C_G(i)$, there exists an element $y \in N_G(T)$, such that $i^{y^2} = j^y = ij$ by [10, 10.22], where $y^3 \in T$. Furthermore $v \in C_G(D)$ and v does not invert T. This implies that $v \in C_G(j) \backslash C_G(j)^\circ$ and $v \in C_G(ij) \backslash C_G(ij)^\circ$. Thus there exist elements $x_1, x_2 \in C_G(D)$ such that $v^y = v x_1$ and $v^{y^2} = v x_2$, where $x_1 \in C_G(j)^\circ$ and $x_2 \in C_G(ij)^\circ$ by Corollary 24.

Let $P := C_T(v)^\circ$. Then $P = \{ l l^v | l \in T \cap L_1 \}$ and $C_T(v) = P\langle i \rangle$. On the other hand x_n either inverts T or centralizes it for some $n = 1, 2$. If x_n would centralize T for $n = 1, 2$, then $P = C_T(v x_n)^\circ = (C_T(v)^{y^n})^\circ = P^{y^n}$. Contradiction as P contains a unique involution from D. Thus x_1 and x_2 invert T.

Let $T[v] := \{ t \in T | t^v = t^{-1} \}$. Then $T[v] \leq T$ and $T[v]^\circ = \{ l l^{-v} | l \in T \cap L_1 \}$. Obviously $T = PT[v]$. As x_n inverts T for $n = 1, 2$, $C_T(x^{y^n}) = C_T(v x_n) = T[v]$. Thus $T[v]^\circ = (C_T(v^y)^\circ)^y = (T[v]^\circ)^y$. Contradiction again, as $T[v]^\circ$ contains a unique involution from D. Hence $C_G(i) = C_G(i)^\circ$ is connected. $\square$

Proposition 27 *If $C_G(i)$ is not connected, then $C_G(i) = C_G(i)^\circ \rtimes \langle u \rangle$, where $u \in C_G(D)$ is an involution such that $L_1^u = L_2$.*

Proof. As $C_G(i)$ is not connected, i and j are not conjugate in G by Proposition 26 and $C := C_G(j)^\circ \cong \mathrm{PSL}_2(K) \times K^*$ by Corollary 19. Let $u \in (N_G(T) \cap C) \backslash T$ be an involution. Then $u \in C_G(D)$ but $u \notin C_G(i)^\circ$, as u does not invert T. Thus $C_G(i) = C_G(i)^\circ K_i \langle u \rangle$ by Lemma 24 and we need to show that $K_i = \langle i \rangle$.

As K_i centralizes $Z(C)$, $K_i \leq K_j C$ by Lemma 3 and Lemma 4. We show that $C_{K_i}(u) = \langle i \rangle$. u acts on K_i. Let $k \in C_{K_i}(u)$. Then k centralizes the elementary abelian subgroup $E_1 := \langle i, j, u \rangle$. As $K_i \leq K_j C$ and $C_C(E_1) = Z(C)\langle u \rangle$, we must have $k \in K_j \langle u \rangle$ by Lemma 23. Thus $k^2 \in K_j \cap K_i = 1$ by Lemma 23 and $k \in \langle i \rangle$ by Lemma 23 again. Especially $C_{K_i}(u) = \langle i \rangle$.

As $|K_i| / |C_{K_i}(u)| = |S|$, where $S = \{ [k, u] | \, k \in K_i \} \subseteq K_i$ consists of elements inverted by u by [10, ex. 17, p. 7], $2|S| = |K_i|$. Let $s \in S$. Then $u^s \in C$ and $u^s = s^u u s = u s^2$. Furthermore s centralizes T and thus $u^s \in N_G(T) \backslash T$ which implies that there exists a $t \in T$ such that $u s^2 = u^s = ut$. Hence $s^2 = t \in K_i \cap T = \langle i \rangle$. Assume that $s^2 = i$. Then $u^s = ui$. As $\langle j, u \rangle$ is

conjugate to D in C, u and uj belong to different conjugacy classes. However, there are at most two conjugacy classes of involutions in $uC_G(i)^\circ$ by elementary computation, namely u^T and $(ui)^T$. Contradiction. Hence $s^2 = 1$ by Lemma 23 and $s \in \langle i \rangle$. Then $|K_i| = 2|S| \leq 4$.

We finally prove that $|S| = 1$. Assume that $|S| = 2$. Then $|K_i| = 4$ and there exists an element $k \in K_i$ of order 4 by Lemma 23. As $C_{K_i}(u) = \langle i \rangle$, $k^u = k^3 = k^{-1}$. Especially $u^k = k^u uk = uk^2 = ui$. Contradiction as before and we are done. □

Proposition 28 *If $C_G(i)$ is not connected, $C_G(j) \cong (\mathrm{GL}_2(K)/\langle -I \rangle) \rtimes \langle w \rangle$ where $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $w \in C_G(D) \cap C_G(i)^\circ$ is an involution that acts as an inverse-transpose automorphism on $\mathrm{GL}_2(K)$, i.e.*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^w = (ad - bc)^{-1} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$.

Proof. As $C_G(i)$ is not connected, $C_G(j)^\circ \cong \mathrm{PSL}_2(K) \times K^* \cong \mathrm{GL}_2(K)/\langle -I \rangle$ by Proposition 26 and Corollary 19. Set $C := C_G(j)^\circ$.

Let $w \in C_G(D) \cap C_G(i)^\circ$ be an involution that inverts T. We show that $C_G(j) = C \rtimes \langle w \rangle$. As w inverts T, $w \notin C$ and $C_G(j) \geq C \rtimes \langle w \rangle$. To prove the other inclusion let $\overline{C} := C/Z(C)$. By Lemma 3 $C_G(j) = C_{C_G(j)}(\overline{C})C$. Let $h \in C_{C_G(j)}(\overline{C})$. Then $T^h \leq TZ(C) = T$ and $h \in N_G(T)$. As $h \in C_G(j)$ and i and ij are not conjugate, $h \in C_G(D) = T\langle u, w \rangle$. Thus, as $T\langle u \rangle \leq C$, $C_G(j) = C\langle w \rangle = C_{C_G(j)}(\overline{C})C$.

As w inverts T, $w \notin C_{C_G(j)}(\overline{C})$. Thus there exists $v \in C_G(D)$ such that $wv \in C_{C_G(j)}(\overline{C})$. w inverts T and $v \in C$, hence wv inverts $Z(C)$. Furthermore $[wv, C] \subseteq Z(C)$ and $C = Z(C) * C_C(wv)$ by [10, ex. 10, p. 98] where $Z(C) \cap C_C(wv) = \langle j \rangle$. Set $H := C_C(wv)^\circ$. As $C \cong \mathrm{PSL}_2(K) \times K^*$, $H \cong PSL_2(K)$ and $C = Z(C) \times H$. We may assume that $v \in H$.

Let $x \in C$. Then there exists $z \in Z(C)$ and $h \in H$ such that $x = zh$. Furthermore $x^{wv} = z^{-1}h$, and $x^w = z^{-1}h^v$. Thus $h^w = h^v$ for all $h \in H$. Set $T_1 = H \cap T$. As w inverts T_1, v has to invert T_1 as well. Hence $v \in I(N_G(T_1) \backslash T_1)$ and after maybe conjugating w with an element from T, we may assume that v corresponds to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This implies the claim. (Compare Section 3.2.) □

Proof of Theorem 1. Let G be as in Theorem 1. If $C_G(i)$ is connected, then $G \cong \mathrm{G}_2(K)$ by Corollary 21. If on the other hand $C_G(i)$ is not connected, $C_G(i)$ and $C_G(j)$ are isomorphic to centralizers of involutions in $\mathrm{PSp}_4(K)$ by Proposition 27 and Proposition 28. Thus $G \cong \mathrm{PSp}_4(K)$ by Theorem 6. □

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